Space-time chaos in the system of weakly interacting hyperbolic systems

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *We consider infinite-dimensional dynamical systems which are lattices of some weakly-interacting hyperbolic systems. We describe the construction of their BRS-measures which are mixing with respect to the group of space-time shifts. The proofs use methods of statistical mechanics.*

1. INTRODUCTION

I. M. Gelfand always taught us that there might be different infinities. In this paper we consider those infinite-dimensional dynamical systems for which all degrees of freedom are in some sense equivalent. The direct consequence of this equivalence is the appearence of a new symmetry group acting on the degrees of freedom by shifts and commuting with the dynamics. Our main problem will be to study ergodic properties of the total group generated by the shifts and the dynamics, i.e. the group of space-time translations. If this group preserves some probability measure and is mixing with respect to it then we shall say that the space-time dynamics exhibits the space-time chaos.

The first example which motivated our investigations was the diffusion-reaction equation of the form

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(1) $$\frac{\partial u}{\partial t} = f(u) + \epsilon \Delta u.$$

The group of space translations acts as the group of shifts $(S^z u)(x) = u(x + z), z \in R^1$ and commutes obviously with the equation (1). However this example is still too difficult for the complete analysis and we shall deal with more simple models. Infinite-dimensional dynamical systems with space-time translations are oftenly encountered in statistical mechanics (see [1]). It is also believed that the fully developped turbulence involves infinitely many degrees of freedom and in principle can be described by systems of a similar type after some normalization.

Our first step consists in discretizing (1) and consideration the transformations of the following type

(2) $$u_{t+1}(x) = f(u_t(x)) + \epsilon(u_t(x+1) + u_t(x-1)),$$

$x \in \mathbb{Z}^\nu$. We shall deal only with the case of $\nu = 1$. The last terms describes the interaction of different degrees of freedom. We shall see that it can be taken in a much more general form. For small ϵ the properties of the mapping corresponding to (2) depend very heavily on the properties of f. In [2] the authors considered the situation when f was an one-dimensional expanding map but the second term in (2) was different. The main result of [2] was the contruction of a natural invariant measure with respect to the group $\mathbb{Z}^2$ generated by the shift and the dynamics which was mixing, i. e. we had the space-time chaos.

The goal of this and subsequent papers is to show a much more general result which states the space-time chaos when is an arbitrary hyperbolic transformation. The corresponding technique for analogous finite- dimensional situations is quite well-known (see [1, 3, 4]). The so-called Markov partitions give a possibility to reduce all arising problems to problems concerning one-dimensional lattice models of statistical mechanics with rapidly decaying interactions. We shall show that the infinite-dimensional mappings defined by (2) can be reduced in some sense to two-dimensional lattice models. The theory of such models is less trivial because there migh be phase-tranitions. The case of small ϵ corresponds to the high-temperature region in statistical mechanics where there are no phase transitions. However it is possible that the increase of ϵ leads to some interesting bifurcations similar to phase transitions. From the point of view of dynamics it can be connected with the widely observed coherent structures in extended systems.

Now we shall gibe more exact definitions.

Denote by M a C^∞ smooth compact p-dimensional Riemannian manifold. Assume that f is a topologically transitive C^2-Anosov diffeomorphism. Its stable and unstable subspaces at any point $x \in M$ are denoted as $E^s_f(x), E^u_f(x)$. Without any loss of generality we may assume that M is equipped with the Liapunov metrics. For any $x \in M$ we have

$$\text{(3)} \qquad \begin{aligned} T_x M &= E^s_f(x) \otimes E^u_f(x), \\ \mathrm{d} f E^{s,u}_f(x) &= E^{s,u}_f(f(x)) \end{aligned}$$

and for every $n > 0$

$$\text{(4)} \qquad \begin{aligned} ||\mathrm{d} f^n v|| &\leq \lambda^n ||v||, \quad v \in E^s_f(x), \\ ||\mathrm{d} f^{-n} v|| &\leq \lambda^n ||v||, \quad v \in E^u_f(x), \end{aligned}$$

where $\lambda, 0 < \lambda < 1$ is a constant not depending on x and n.

Consider now infinitely many copies of M and f i.e. $M_i \equiv M, f_i \equiv f, i \in \mathbf{Z}$. Put

$$\mathcal{M} = \bigotimes_{i \in \mathbf{Z}} M_i, \ \mathcal{F} = \bigotimes_{i \in \mathbf{Z}} f_i.$$

$\mathcal{M}$ is an infinite-dimensional Banch manifold with the norm $||x|| = \sup_{i \in \mathbf{Z}} ||x_i||$, $x = (x_i)$, $x_i \in TM_i$, $\mathcal{F} \in C^2$ and is a topologically transitive Anosov diffeomorphism for which stable and unstable subspaces at $x \in M$ have the form

$$\mathcal{E}^{s,u}_{\mathcal{F}}(x) = \bigotimes_{i \in \mathbf{Z}} E^{s,u}_{f_i}(x_i)$$

and satisfy (3) and (4). It is natural to call $\mathcal{F}$ as a chain of non-interacting Anosov diffeomorphisms.

For any $m, n \in \mathbf{Z}, m \leq n$ denote

$$M_{m,n} = \bigotimes_{i=m}^{n} M_i, \ F_{m,n} = \bigotimes_{i=m}^{n} f_i$$

and $P_{m,n} : \mathcal{M} \to M_{m,n}$ is the natural projection, $P_m = P_{-m,m}$.

Now we shall introduce the interaction. Let g_0 be a C^2-map of $\mathcal{M}$ onto M_0.

DEFINITION 1. g_0 is short-ranged if for some C_0, $0 < C_0 < \infty$ and $x_0, 0 < x_0 < 1$ there exists a sequence of C^2-maps $g_0^{(n)}$ of $M_{-n,n}$ onto $M_0, n = 0, 1, 2, \ldots$ such that $g_0^{(0)} = Id$, $\mathrm{dist}_{C^2_n}(g_0^{(n)}, g_0^{(n-1)}) \leq C_0 \ x_0^n, \mathrm{dist}_{C^2_n}(g_0, g_0^{(n)} \circ P_n) \leq C_0 \ x_0^n$.

In the last inequalities $g_0^{(n)}$, $g_0^{(n-1)}$ are considered as maps of $M_{-n,n}$ onto M_0, $\mathrm{dist}_{C^2_n}$ is the distance in the C^2-topology.

If $g_0 = g_0^{(n)}$ for some n then we shall say that g_0 is an interaction of the finite range. The smallest n is called the radius of interaction. In what follows we shall consider only short-ranged interaction not mentioning this again. Let S be the shift to the right, i.e. $(Sx)_i = x_{i+1}$ where $x = (x_i)$. Put $g_i = S^{-i} g_0 S^i, g_i^{(n)} = S^{-i} g_0^{(n)} S^i$, they map $\mathcal{M}$ (or $M_{-n+i,n+i}$) onto M_i and is also short-ranged. Now we define the interaction map g which acts by the formula

$$g(x) = g(\dots, x_k, \dots, x_{-1}, x_0, x_1, \dots, x_n, \dots) = \\ = (\dots g_{-k}(x), \dots, g_{-1}(x), g_1(x), \dots, g_n(x), \dots).$$

The form g resembles in many respects maps which are encountered in the theory of cellular automata.

We shall study in this paper maps $\Phi = g \circ \mathcal{F}$ which can be called chains of interacting Anosov maps. For such maps we shall construct natural measures invariant under the action of the group $\mathbb{Z}^2$ generated by Φ and S provided that C_0, x_0 are sufficiently small. These measures are analogous of BRS-measures in the theory of finite-dimensional hyperbolic systems (see [1]).

2. PROPERTIES OF STABLE AND UNSTABLE MANIFOLDS OF Φ

The map Φ has natural finite-dimensional approximations. Namely, put $\Phi_{m,n} = g_{m,n} \circ \mathcal{F}_{m,n}$ for any $m, n, m \leq n$ and $n - m$ is even, where

$$g_{m,n}(x_m, \dots, x_n) = \left(g_m^{(0)} x_m, g_{m+1}^{(1)} x_{m+1}, \dots, g_{\frac{n+m}{2}}^{\frac{(n-m)}{2}} x_{\frac{n+m}{2}}, \dots, g_n^{(0)} x_n \right).$$

PROPOSITION 1. *For sufficiently small C_0, x_0 the maps $\Phi, \Phi_{m,n}$ are transitive C^2-Anosov diffeomorphisms.* ∎

REMARK. One can show that $\Phi, \Phi_{m,n}$ are topologically conjugate with $\mathcal{F}, \mathcal{F}_{m,n}$ respectively with the help of homeomorphisms which are close to identity, in the last case uniformly over m, n.

Denote by $\mathcal{E}_{\Phi}^{s,u}(y), \mathcal{E}_{\Phi_{m,n}}^{s,u}(y)$ the stable and unstable subspaces for the mappings Φ and $\Phi_{m,n}$. Also ρ^* is the metrics in the tangent space induced by the metrics ρ on $\mathcal{M}$ where $\rho(x, y) = \sup_i \rho_i(x_i, y_i)$ and ρ_i is the Riemannian distance on M_i.

PROPOSITION 2. *For any $y \in \mathcal{M}$*

$$\rho^* \left(\mathcal{E}_{\Phi}^{s,u}(y), \mathcal{E}_{\mathcal{F}}^{s,u}(y) \right) \leq \text{const} \cdot x_0;$$

for any $y \in M_{m,n}$

$$\rho^*\left(\mathcal{E}^{s,u}_{\Phi_{m,n}}(y), \mathcal{E}^{s,u}_{\mathcal{F}_{m,n}}(y)\right) \leq \text{const} \cdot x_0 . \quad \blacksquare$$

Our first essential result concerns the structure of stable and unstable manifolds of Φ and $\Phi_{m,n}$. The formal theory gives the following. Denote by $B^{s,u}_\delta(y), B^{s,u}_{m,n,\delta}(y)$ the balls of radius δ centered at the origin in the subspaces $\mathcal{E}^{s,u}_{\mathcal{F}}(y), \mathcal{E}^{s,u}_{\mathcal{F}_{m,n}}(y)$ respectively.

PROPOSITION 3. *For some* $\delta = \delta(C_0, x_0) > 0, \lambda_1, \lambda < \lambda_1 < 1$

1) for any $y \in \mathcal{M}$ *there exists the one-to-one mappings* $\varphi^{s,u}(y) : B^{s,u}_\delta(y) \to \mathcal{M}$ *such that* $V^s_\delta(y) = \varphi^s(y)\, B^s_\delta(y), V^u_\delta(y) = \varphi^u(y)\, B^u_\delta(y)$ *are local stable and unstable manifolds of the point* y;

2) for each $m, n,\ m \leq n$ *and any* $y \in M_{m,n}$ *there exist the one-to-one mappings* $\varphi^{s,u}_{m,n}(y) : B^{s,u}_{m,n,\delta}(y) \to M_{m,n}$ *such that* $V^s_{m,n,\delta}(y) = \varphi^s_{m,n}(y)\, B^s_{m,n,\delta}(y), V^u_{m,n,\delta}(y) = \varphi^u_{m,n}(y)\, B^u_{m,n,\delta}(y)$ *are local stable und unstable manifolds of the point* y;

3) $\varphi^{s,u}(y)(0) = y, \varphi^{s,u}_{m,n}(y)(0) = y$;

4)

$$\rho\left(\Phi^k y, \Phi^k z\right) \leq \lambda_1^k \rho(y,z) \text{ for } k \geq 0, \quad z \in V^s_\delta(y);$$

$$\rho\left(\Phi^{-k} y, \Phi^{-k} z\right) \leq \lambda_1^k \rho(y,z) \text{ for } k \geq 0, \quad z \in V^u_\delta(y);$$

$$\rho\left(\Phi^k_{m,n} y, \Phi^k_{m,n} z\right) \leq \lambda_1^k \rho(y,z) \text{ for } k \geq 0, \quad z \in V^s_{m,n,\delta}(y);$$

$$\rho\left(\Phi^{-k}_{m,n} y, \Phi^{-k}_{m,n} z\right) \leq \lambda_1^k \rho(y,z) \text{ for } k \geq 0, \quad z \in V^u_{m,n,\delta}(y). \blacksquare$$

The next lemma describes the first important property of $V^{s,u}_\delta(y)$ which apparently is valid only for small C_0, x_0.

LEMMA 1. *For small enough* C_0, x_0 *there exists a constant* C_1 *such that*

1) fix $i, k \in \mathbb{Z}$ *and take* $y \in \mathcal{M}$ *and arbitrary* $u', u'' \in B^{s,u}_\delta(y)$ *for which* $u'_j = u''_j$ *for all* $j \neq k$ *then*

$$\rho_i\left((\varphi^{s,u}(y)(u'))_i, \quad (\varphi^{s,u}(y)(u''))_i\right) \leq C_1 x_0^{|i-k|}\, \rho_k(u'_k, u''_k);$$

$$\text{dist}(\mathrm{d}\,\varphi^{s,u}(y)(u')_i, \quad \mathrm{d}\,\varphi^{s,u}(y)(u'')_i) \leq C_1 x_0^{|i-k|} \rho_k(u'_k, u''_k);$$

2) take $m, n, m \leq n$, fix $i, k, m \leq i, k \leq n$ *and take* $y \in M_{m,n}, u', u' \in B^{s,u}_{m,n,\delta}(y)$ *for which* $u'_j = u''_j$ *for all* $j \neq k$; *then*

$$\rho_i((\varphi^{s,u}_{m,n}(y)(u'))_i, \quad (\varphi^{s,u}_{m,n}(y)(u''))_i) \leq C_1 x_0^{|i-k|}\, \rho_k(u'_k, u''_k);$$

$$\text{dist}(\mathrm{d}\,\varphi^{s,u}_{m,n}(y)(u')_i, \quad \mathrm{d}\,\varphi^{s,u}_{m,n}(y)(u'')_i) \leq C_1 x_0^{|i-k|} \rho_k(u'_k, u''_k). \quad \blacksquare$$

This lemma and the previous propositions show that each local manifold $V_\delta^s(y)$, $V_\delta^u(y)$ has a natural structure of the direct product. The mappings $\varphi^{s,u}(y)$ introduce this structure. Lemma 1 shows that if we consider the i-th coordinate $(\varphi^{s,u}(y)(u))_i$ as a mapping of $B_\delta^{s,u}(y)$ into M_i then it is short-ranged in the same meaning as the short-range of the interaction. On should also remark that the interaction might have a finite radius of interaction but $\varphi^{s,u}(y)$ will not have this property. The same is true for $\varphi_{m,n}^{s,u}(y)$.

Further we shall deal with functions, mappings etc. which depend very weakly on variables whose number is far from some fixed number $k \in \mathbf{Z}$. In order to treat such objects we need a new metrics. Namely, fix $x_1, 0 < x_1 < 1$ and take $y', y'' \in M_{m,n}$.

$$\rho_k(y', y'') = \max_{m \le i \le n} \rho(y_i', y_i'') x_i^{-|i-k|}.$$

LEMMA 2. *For sufficiently small x_0 there exists a constant C_2 such that for any $y \in M_{m,n}$ and $l \ge 0$*

$$\rho_k\left(\Phi_{m,n}^{-l} y, \Phi_{m,n}^{-l} z\right) \le C_2 \lambda_1^l \rho_k(y, z), \qquad z \in V_{m,n,\delta}^u(y);$$
$$\rho_k\left(\Phi_{m,n}^{l} y, \Phi_{m,n}^{l} z\right) \le C_2 \lambda_1^l \rho_k(y, z), \qquad z \in V_{m,n,\delta}^s(y). \qquad \blacksquare$$

3. CONSTRUCTION OF NATURAL MEASURES ON LOCAL MANIFOLDS

In the finite-dimensional situtations the first step which gives BRS-measures is the constructions of the measures on local stable and unstable manifolds which are limits of the images of the Lebesgue measures. In our case these local manifolds are infinite-dimensional. However the natural measures also exist but they are decribed as a non-homogeneous Gibbs states.

Fix $m, n, m \le n, y \in M_{m,n}$. For any $z \in V_{m,n,\delta}^u(y)$ and $l \ge 0$ consider the expression

$$r_{m,n}(z, y, l) = \prod_{i=0}^{l} \mathcal{I}(\Phi_{m,n}^{-i} z)\ [\mathcal{I}^u(\Phi_{m,n}^{-i} y)]^{-1}$$

where $\mathcal{I}^u(\Phi_{m,n}^{-1} z)$ is the Jacobian of the mapping

$$\Phi_{m,n}^{-1} : \Phi_{m,n}^{-i} V_{m,n,\delta}^u(y) \to \Phi_{m,n}^{-i-1} V_{m,n,\delta}^u(y).$$

It is easy to show that there exists the limit

$$r_{m,n}(z, y) = \lim_{l \to \infty} r_{m,n}(z, y, l).$$

The limiting function is uniformly continuous in $y \in M_{m,n}$ and $z \in V^u_{m,n,\delta}(y)$ and satisfies the homologous equation

$$r_{m,n}(z,y) \cdot r_{m,n}(w,z) = r_{m,n}(w,y).$$

Introduce the probability measure $\mu_{m,n}(\cdot\,; y)$ on $B^u_{m,n,\delta}(y)$ by putting

$$\mathrm{d}\,\mu_{m,n}(z;y) = \frac{1}{Z_{m,n}(y)}\, r_{m,n}(z,y)\pi_{m,n}(z)\mathrm{d}\,z_m \cdot \ldots \cdot \mathrm{d}\,z_n.$$

Here $z_m, \ldots, z_n$ are natural coordinates on $B^u_{m,n,\delta}(y)$; $\pi_{m,n}(z)$ is the Jaconian which comes from the mapping $\varphi^u_{m,n}(y) : B^u_{m,n,\delta}(y) \to V^u_{m,n,\delta}(y)$ in such a way that $\pi_{m,n}(z)$ $\mathrm{d}\,z_m \cdot \ldots \cdot \mathrm{d}\,z_n$ is the differential of the Riemannian volume on $V^u_{m,n,\delta}(y)$; $Z_{m,n}(y)$ is the normalization factor. It is an analogy of the partition function in statistical physics. Our next problem is to study the limit of measures $\mu_{m,n}(\cdot\,; y)$ as $m \to -\infty, n \to \infty$.

The point y is now fixed and we shall not denote the dependence on y explicitly. Put $P_{m,n}(z_m, \ldots, z_n) = (Z_{m,n}(y))^{-1} r_{m,n}(z,y)\pi_{m,n}(z)$ and

$$\tilde{P}_{m,n}(z_m, \ldots, z_k) = \int P_{m,n}(z_m, \ldots, z_n)\mathrm{d}\,z_{k+1} \cdot \ldots \cdot \mathrm{d}\,z_n,$$

$$\tilde{P}_{m,n}(z_k | z_{k-1}, \ldots, z_m) = \frac{\tilde{P}_{m,n}(z_m, \ldots, z_k)}{\tilde{P}_{m,n}(z_m, \ldots, z_{k-1})}.$$

LEMMA 3. *For sufficiently small* C_0, x_0 *there exist* $\gamma_1, 0 < \gamma_1 < 1, C_3 > 0$ *not depending on* m, n *and such that*

$$\exp\{-C_3\gamma_1^{k-j}\} \le \min_{z'_j, z''_j} \frac{\tilde{P}_{m,n}(z_k | z_{k-1}, \ldots, z'_j, \ldots, z_m)}{\tilde{P}_{m,n}(z_k | z_{k-1}, \ldots, z''_j, \ldots, z_m)} \le$$
$$\le \max_{z'_j, z''_j} \frac{\tilde{P}_{m,n}(z_k | z_{k-1}, \ldots, z'_j, \ldots, z_m)}{\tilde{P}_{m,n}(z_k | z_{k-1}, \ldots, z''_j, \ldots, z_m)} \le \exp\{C_3\gamma_1^{k-j}\}.$$

The proof of this lemma is based upon the estimation given in the lemmas 1, 2. It also shows that one can take $\gamma_1 \to 0$, $C_3 \to 0$ as $x_0 \to 0$, $C_0 \to 0$. We are encountered now with functions depending on many variables but the dependence on variables which have numbers far from some fixed number decays quickly with the distance from this number. We shall need a special definition for this.

DEFINITION 2. Let be given $k, m \le k \le n$, $\gamma \in (0,1)$. Then $\mathcal{D}_k(\gamma)$ is the space of continuous functions $h(z)$, $z \in B^u_{m,n,\delta}(y)$ such that

$$\max |h(z') - h(z'')| \le C(h)\gamma^{|j-k|}.$$

Maximum is taken over all $z' = (z'_i)$, $z'' = (z''_i)$ for which $z'_i = z''_i$ $i = m, \dots, n$ for all $i \neq j$. Infimum of all $C(h)$ is denoted by $||h||_\gamma$.

We shall deal also with positive functions $h(z)$ for which $\ln h(z) \in \mathcal{D}_k(\gamma)$. The space of such functions is denoted by $\mathcal{D}_k^0(\gamma)$. For $h \in \mathcal{D}_k^0(\gamma)$

$$||h||_\gamma = \inf_C \left\{ C : \exp\{-C\gamma^{|k-j|}\} \le \min \frac{h(z')}{h(z'')} \le \right.$$
$$\left. \le \max \frac{h(z')}{h(z'')} \le \exp\{-C\gamma^{|k-j|}\} \right\}$$

where min and max are taken over such pairs z', z'' that z'_i, z''_i for all $i \neq j$. Lemma 3 implies that $\tilde{P}_{m,n}(z_k | z_{k-1}, \dots, z_m) \in \mathcal{D}_k^0(\gamma_1)$.

Now take a function $h(z_m, \dots, z_k) \in \mathcal{D}_k(\gamma)$ and introduce the stochastic operator

$$(Q_k h)(z_m, \dots, z_{k-1}) = \int h(z_m, \dots, z_{k-1}, z)\, P_{m,n}(z | z_{k-1}, \dots, z_m) \mathrm{d}\, z.$$

LEMMA 4. *Under the conditions of lemma 3 there exist* $\gamma_2 \in (\gamma_1, 1)$ *and* $\lambda_3 \in (0, 1)$ *such that for any*

$$h(z_m, \dots, z_k) = h \in \mathcal{D}_k(\gamma_2)$$
$$Q_k h \in \mathcal{D}_{k-1}(\gamma_2), \qquad ||Q_k h||_{\gamma_2} \le \lambda_3 ||h||_{\gamma_2}$$

Proof. We have

$$(Q_k h)(z_m, \dots, z'_j, \dots, z_{k-1}) - (Q_k h)(z_m, \dots, z''_j, \dots, z_{k-1}) =$$
$$= \int [h(z_m, \dots, z'_j, \dots, z_{k-1}, z) - h(z_m, \dots, z''_j, \dots, z_{k-1}, z)] \cdot$$
$$\cdot P_{m,n}(z | z_{k-1}, \dots, z'_j, \dots, z_m) \mathrm{d}\, z +$$
$$+ \int h(z_m, \dots, z''_j, \dots, z_{k-1}, z) P_{m,n}(z | z_{k-1}, \dots, z'_j, \dots, z_m) \cdot$$
$$\cdot \left[1 + \frac{P_{m,n}(z | z_{k-1}, \dots, z''_j, \dots, z_m)}{P_{m,n}(z | z_{k-1}, \dots, z'_j, \dots, z_m)} \right].$$

Hence

$$|(Q_k h)(z_m, \dots, z'_j, \dots, z_{k-1}) - (Q_k h)(z_m, \dots, z''_j, \dots, z_{k-1})| \le$$
$$\le C(h)\gamma_2^{k-j} + C(h) C_3 \gamma_1^{k-j} =$$
$$= C(h)\gamma_2^{k-j-1}(\gamma_2 + C_3\gamma_1).$$

Putting $\lambda_3 = \gamma_2 + C_3\gamma_1$ we get the statement of lemma. ■

Now we can complete the construction of measures on the local unstable manifolds. Take any $y \in \mathcal{M}, y = \{\dots, y_m, \dots, y_n, \dots\}, y_{m,n} = P_{m,n}y$ and construct the balls $B^u_{m,n,\delta}(y), B^u_\delta(y_{m,n})$. We may assume that $B_{m,n,\delta}(y) \subset B^u_\delta(y_{m,n})$. Take any semi-infinite sequence $(\dots z_m, \dots, z_k)$.

LEMMA 5. *There exists the limit*

$$P(z_k|z_{k-1},\dots) \lim_{\substack{m\to-\infty\\ n\to\infty}} P_{m,n}(z_k|z_{k-1},\dots,z_m).$$

The limiting function is a probability density and belongs to the space $\mathcal{D}^\circ(\gamma_2)$.

For proving the lemma we follow for changes of densities $P_{m,n}$ when we pass from m to $m-1$ and from n to $n+1$. In the first case the change is less then const $\cdot\gamma_2^{k-m}$. It follows directly from Lemma 3. The passage $n \to n+1$ is slightly more difficult. Firstly we integrate over the variable $n+1$ and get a function $g(z_m,\dots,z_n) \in \mathcal{D}_n(\gamma_2)$. Then we apply Lemma 4 which shows that after $n-k$ integrations of g over $\mathrm{d}\,z_n, \dots, \mathrm{d}\,z_{k+1}$ we get a function which is exponentially close to a constant. But this constant is equal to one because the integral of the function g is equal to one. Now we formulate the final result.

THEOREM 1. *Take any* $B^u_\delta(y) = \bigotimes_{i\in\mathbb{Z}} B^u_\delta(y_i), y = \{\dots, y_m, \dots, y_n, \dots\}$. *The probability measures* $\mu_{m,n}$ *converge weakly as* $m \to -\infty, n \to \infty$ *to a probability measure* $\mu_{B^u_\delta(y)}$. *defined the Borel* δ*-algebra of the space of the space* $B^u_\delta(y)$. ■

The one-sided conditional probabilities of this measure are equal to the limiting densities given in Lemma 5.

In the next section we discuss the problem how to construct the measure on $\mathcal{M}$ compatible with the measures $\mu_{B^u_\delta(y)}$.

4. THE CONSTRUCTION OF THE BRS-MEASURE

As in the finite-dimensional situation the natural way to construct the BRS- measure is to take the shifts $(\Phi^r)^*\mu_{B^u_\delta(y)}$ are consider their weak limit. However it is not so simple due to the infinite dimension of the phase space. Firstly we remark that the measures $\mu_{B^u_\delta(y)}$ are compatible in the following sense. Take $y_1 \in V^u_\delta(y)$ and consider the intersection. $\varphi^u(y)(B^u_\delta(y)) \cap \varphi^u(y_1)(B^u_\delta(y_1))$. Then the restrictions of $\mu_{B^u_\delta(y)}, \mu_{B^u_\delta(y_1)}$ to this intersection coincide up to a constant factor. Thus we have on each total unstable manifold a family of compatible measures.

Construct a partition ζ which is Markov with respect to the map Φ and invariant under S. As in the case of finite-dimensional dynamical systems one can construct first some Marcov cover (cf. [1, 3, 4]). The Markov partition ζ determines the symbolic representation of our $\mathbb{Z}^2$-dynamical system. As in [2] the methods of statistical mechanics give a posibility to show that there exist only one invariant measure which produces conditional measures on local layers constructed above. Detailed proofs of all formulated results will be published elsewhere.

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